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On the behaviour of the laminar boundary-layer equations of mixed convection near a point of zero skin friction

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The boundary-layer equations of mixed convection are examined in the vicinity of separation. The correlation between the uniform wall temperature case and that of compressible boundary layer flow is outlined. Goldstein–Stewartson–Buckmaster theory is thus appropriate and associated indeterminacies in the theory are evaluated from a numerical integration. The case of uniform heat flux at the wall is then examined theoretically. Significantly it is concluded that the original Goldstein–Stewartson theory is sufficient to describe the structure of the singularity at separation in this case. Indeterminacies associated with the theory are determined via a reconciliation between analytical and numerical representation of skin friction and heat transfer coefficients near separation.

1. Introduction

The foundations for the appreciation of the behaviour of the laminar incompressible boundary-layer equations at a point x_s of vanishing skin friction were laid down in the classic paper of Goldstein (1930). It was not until Goldstein (1948), however, that a level of accord between analysis and numerical integration of the governing momentum equation was demonstrated. This was achieved by developing the tentative analysis of 1930 on the assumption that the first compatibility condition for the absence of singularities was satisfied. As a result theoretical and numerical evidence of skin friction behaviour as $(x_s-x)^{\frac{1}{2}}$ was reconciled. Anomalies in the analysis associated with the requirement of algebraic behaviour at large η for coefficient functions in the Goldstein expansion were settled by Stewartson (1958). Here $\eta = y'/2^{\frac{1}{2}}(x_s-x)^{\frac{1}{4}}$, where y' is a dimensionless distance measured normal to the wall. Further work by Terrill (1960) confirmed the validity of the Stewartson modifications and consequently the structure about the singularity in the incompressible case is regarded as fully understood.

The structure about the singularity in flows governed by the coupled boundary-layer equations of momentum and energy has up to now proved less tractable. Discussion of two relevant circumstances have appeared in the literature, namely separation in compressible boundary layer flow and separation in mixed convection flow. The former case was first examined from a theoretical standpoint by Stewartson (1962). Following an analysis closely patterned on his earlier work on the incompressible case he was led to the conclusion that a general compressible laminar boundary layer can develop a singularity at a point of zero skin friction only if the heat transfer at that point is also zero. At variance with this conclusion was subsequent unpublished numerical evidence of singular behaviour at separation (private communication from

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P. G. Williams, University College London). This anomaly was ultimately resolved by Buckmaster (1970) who demonstrated a complicated but self-consistent expansion involving new logarithmic terms and their products which generated a skin friction representation vanishing as $(x_s - x)^{\frac{1}{2}} \ln (x_s - x)$. More recently Davies & Walker (1977) have undertaken a thorough numerical investigation of compressible boundary layer separation. Despite some slight reservations over the accuracy of their results in the immediate vicinity of separation it does appear that the Goldstein–Stewartson–Buckmaster theory satisfactorily accounts for the skin friction behaviour for both hot and cold walls.

Separation in mixed convection, on the other hand, was first discussed by Merkin (1969), who examined the effect of opposing buoyancy forces on the boundary layer flow over a uniform temperature semi-infinite vertical flat plate in a uniform stream. His numerical evidence was indicative of a square root singularity at separation. Moreover an analytic formulation, appropriate at separation in this context, yields equations which almost exactly coincide with those first addressed by Stewartson (1962). Thus Merkin's results in fact provided the first reported contradiction of Stewartson's original conjecture. Naturally these results should therefore be compatible with the Buckmaster theory and its associated expansions. By amending the uniform temperature constraint to that of a uniform heat flux at the plate Wilks (1974) sought to provide additional information concerning circumstances involving irregularities at a point of zero skin friction. Preliminary examination of the results suggested, surprisingly, the presence of a three fifths singularity at separation. Subsequent computations (Davies & Walker) have indicated the sensitivity of the numerical scheme to the form of modelling of the uniform heat flux boundary condition. When this is accounted for the familiar square root behaviour is recovered. No theoretical study of these latter circumstances has as yet been reported.

From the preceding discussion it may be conjectured that recourse to Buckmaster forms of expansion is in some sense a property of the coupling of the governing momentum and energy equations. Is it inevitable that such a complicated structure will be required to account for irregularities encountered in numerical integration to separation in such circumstances? In the work that follows we suggest that this need not necessarily be the case. We are led to this conclusion on the basis of an analytic examination about the separation occurring in the uniform heat flux problem. Significantly, it is indicated that for this problem a Goldstein–Stewartson expansion will apparently suffice so long as it is again assumed, as in the incompressible case, that the first condition for the absence of singularities is satisfied. No complications involving non-terminating sequences of coefficient functions are encountered and progress may hence be made towards more explicit representations of the higher order coefficient functions.

In order to substantiate the above conclusion a further numerical solution of the mixed convection separation problems was undertaken, employing an alternative method of solution to that used previously. A distinction in the nature of the singularities in the two problems of uniform temperature and uniform heat flux was immediately apparent. The numerical evidence clearly indicated that the singularity associated with the uniform heat flux integration was of a less complicated nature than that occurring in the constant temperature integration. This is indeed only to be expected if the theoretical predictions are valid. An indeterminacy appearing in the theoretical representations of skin friction and heat transfer coefficients may be estimated either through a comparison with the numerical values of skin friction coefficient or through a comparison with the numerical values of heat transfer coefficient. The level of agreement between the two independent estimates of this indeterminacy is taken as confirmation that the structure at separation for the constant heat flux problem has been satisfactorily accounted for by the Goldstein–Stewartson expansions.

2. The equations of mixed convection

The flows envisaged in this paper involve the flow of a uniform stream U along a semi-infinite flat plate extending vertically downwards with its leading edge horizontal. Heat is supplied to the flow by diffusion and convection from the plate either as a result of (a) a uniform temperature T_1 at the plate or (b) a uniform heat flux q from the surface. This heating, relative to the surrounding ambient temperature T_0, generates buoyancy forces which oppose the free stream and ultimately lead to separation of the boundary layer formed from the leading edge. The governing equations, incorporating the assumptions that changes in density are significant only in producing buoyancy forces and that viscous dissipation may be neglected, are

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = -g\beta(T-T_0)+\nu\frac{\partial^2 u}{\partial y^2}, \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y} = \kappa\frac{\partial^2 T}{\partial y^2}. \tag{3}$$

Here u and v are velocity components associated with increasing x and y respectively, where x measures distance along the plate from the leading edge $x = 0$ and y is measured normally outwards from the plate; T is the temperature of the fluid and g the acceleration due to gravity, β the coefficient of thermal expansion, κ the thermometric conductivity and ν the kinematic viscosity are all taken as constant. Solution of (1)–(3) is required subject to boundary conditions

$$\left.\begin{array}{l} u = v = 0 \quad \text{on} \quad y = 0, \\ u \to U, \quad T \to T_0 \quad \text{as} \quad y \to \infty, \\ u = U, \quad T = T_0 \quad \text{at} \quad x = 0; \end{array}\right\} \tag{4}$$

$$\left.\begin{array}{ll} (a) & T = T_1 \\ (b) & \dfrac{\partial T}{\partial y} = \dfrac{-q}{k} \end{array}\right\} \quad \text{on} \quad y = 0,$$

where k is thermal conductivity.

Merkin (1969) and Wilks (1974) have demonstrated the relevance of characterizing non-dimensional coordinates in formulating each of the problems under discussion, namely

$$(a) \quad \xi = \frac{g\beta(T-T_0)\,x}{U^2} = \frac{x}{l_M}, \tag{5}$$

$$(b) \quad \tilde{\xi} = \left(\frac{2^3 g^2\beta^2 q^2 \nu}{5^2 k^2 U^5}\right)^{\frac{1}{3}} x = \frac{x}{l_W}. \tag{6}$$

Each co-ordinate reflects the local relative importance of viscous and buoyancy forces. Near the leading edge the dominant feature of the flows, is the viscous retardation of the free stream U. Accordingly, transformations which render equations (1)–(3) amenable to numerical integration are

$$\left.\begin{aligned}&(a)\quad \psi = (2\nu Ux)^{\frac{1}{2}}\bar{f}(\xi,\zeta), &&(b)\quad \psi = (2\nu Ux)^{\frac{1}{2}}\tilde{f}(\tilde{\xi},\zeta),\\ &T-T_0 = (T_1-T_0)\,\bar{\theta}(\xi,\zeta), &&T-T_0 = \frac{-q}{k}\left(\frac{2\nu x}{U}\right)^{\frac{1}{2}}\tilde{\theta}(\tilde{\xi},\zeta),\end{aligned}\right\} \tag{7}$$

where $\zeta = y(U/2\nu x)^{\frac{1}{2}}$. For brevity the associated equations and boundary conditions are omitted – they are readily available in the references cited. The transformations are quoted, however, to clarify certain correlations which are later required in assessing compatibility between numerical results and theory.

To examine the behaviour near separation, equations (1)–(3) are first non-dimensionalized and then transformed in a manner analogous to that of Goldstein (1948). Taking

$$x' = \frac{(x_s-x)}{l},\quad y' = \frac{R^{\frac{1}{2}}y}{l},\quad \psi' = \frac{R^{\frac{1}{2}}\psi}{lU},\quad u' = \frac{u}{U},\quad v' = \frac{R^{\frac{1}{2}}v}{U}$$

together with

$$(a)\quad \theta' = \frac{T-T_0}{T_1-T_0},\quad (b)\quad \theta' = \frac{T-T_0 . R^{\frac{1}{2}}}{-q/k . l}, \tag{8}$$

leads to the non-dimensional equations

$$-\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'} = 0, \tag{9}$$

$$-u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'} = \mp\theta'+\frac{\partial^2 u'}{\partial y'^2}, \tag{10}$$

$$-u'\frac{\partial \theta'}{\partial x'}+v'\frac{\partial \theta'}{\partial y'} = \frac{1}{Pr}\frac{\partial^2\theta'}{\partial y'^2} \tag{11}$$

where Pr is the Prandtl number ν/κ, R is the Reynolds number Ul/ν and l is interpreted as follows:

for (a) $\quad l \equiv l_M$ and the minus sign persists in equation (10),

for (b) $\quad l \equiv \frac{2}{5^{\frac{2}{3}}}l_W$ and the plus sign persists in equation (10).

The transformations appropriate to an initial profile displaying a double zero at the origin are

$$\xi = (x')^{\frac{1}{4}},\quad \eta = \frac{y'}{2^{\frac{1}{4}}\xi},\quad \psi' = 2^{\frac{3}{4}}\xi^3 f(\xi,\eta),\quad \theta' = \theta(\xi,\eta). \tag{12}$$

The resulting equations are

$$\frac{\partial^3 f}{\partial\eta^3}-3f\frac{\partial^2 f}{\partial\eta^2}+2\left(\frac{\partial f}{\partial\eta}\right)^2+\xi\left\{\frac{\partial^2 f}{\partial\xi\,\partial\eta}\frac{\partial f}{\partial\eta}-\frac{\partial^2 f}{\partial\eta^2}\frac{\partial f}{\partial\xi}\right\}\mp\theta = 0, \tag{13}$$

$$\frac{1}{Pr}\frac{\partial^2\theta}{\partial\eta^2}-3f\frac{\partial\theta}{\partial\eta}+\xi\left\{\frac{\partial f}{\partial\eta}\frac{\partial\theta}{\partial\xi}-\frac{\partial f}{\partial\xi}\frac{\partial\theta}{\partial\eta}\right\} = 0, \tag{14}$$

where once again $-$ and $+$ of (13) refers to (*a*) and (*b*) respectively. In keeping with previous work on this topic we shall restrict ourselves to the case $Pr = 1$. Boundary conditions play an important role in subsequent developments and their discussion is delayed to later paragraphs.

3. The constant-temperature case (*a*)

Although Stewartson's resolution of the behaviour at incompressible separation invalidated Goldstein's original assumption of a power series representation of the separation velocity profile, the non-analyticity is only introduced in higher order terms. Accordingly it is still helpful, as a preliminary boundary condition, to require that the solution of the boundary layer equations evolve in some sense towards the separation profiles of velocity and temperature expressed as series, and to deal with anomalies as and when they arise. In non-dimensional terms this implies profiles satisfying $(\partial u'/\partial y')_{y'=0} = 0$ and $(\theta')_{y'=0} = b_0 = 1$ at $x' = 0$, namely

$$u' = a_2 y'^2 + a_3 y'^3 + a_4 y'^4 + \ldots, \tag{15}$$

$$\theta' = 1 + b_1 y' + b_2 y'^2 + b_3 y'^3 + \ldots. \tag{16}$$

Note that the first condition for the absence of a singularity in the solution of (10) at $x' = 0$ is then

$$2a_2 - 1 = 0. \tag{17}$$

In view of (15) and (16) it is natural, as a first attempt, to seek series solutions of (13) and (14) in integral powers of ξ,

$$f(\xi, \eta) = \sum_{n=0}^{\infty} f_n(\eta)\, \xi^n, \tag{18}$$

$$\theta(\xi, \eta) = \sum_{n=0}^{\infty} \theta_n(\eta)\, \xi^n \quad (n = 0, 1, 2, \ldots), \tag{19}$$

and require coefficient functions f_n, θ_n to behave algebraically at large η, i.e. expect that

$$\lim_{\eta\to\infty} \frac{f'_n}{\eta^{n+2}} = 2^{\frac{1}{4}n} a_{n+2}, \quad \lim_{\eta\to\infty} \frac{\theta_n}{\eta^n} = 2^{\frac{1}{4}n} b_n, \tag{20}$$

where, from now on, the prime will imply $d/d\eta$. Remaining boundary conditions reflecting impermeability, no slip and uniform temperature at the plate are simply

$$f_n(0) = f'_n(0) = 0; \quad \theta_0(0) = 1; \quad \theta_n(0) = 0 \quad (n \geqslant 1). \tag{21}$$

The equations for f_0, θ_0 now coincide with those of Stewartson (1962) for his f_0, g_0 except that $1 + g_0$ is replaced by θ_0 in the momentum equation.

The equivalence with a particular case of Stewartson's work is complete, as far as further developments are concerned, when it is recognized that $\theta_0 \equiv 1$ is the only acceptable solution of the energy equation under (21). The solution for f_0 with a double zero at the origin and *satisfying the first compatibility condition* (17) is then $f_0 = \frac{1}{6}\eta^3$. The discussion for higher order terms must now exactly parallel that of Buckmaster (1970). The arguments demonstrate the coefficient functions (18) and (19) as incomplete and indicate the need to introduce coefficient functions $f_n(\xi, \eta)$ and $\theta_n(\xi, \eta)$ $(n \geqslant 1)$

whose ξ dependence is logarithmic. The development of the representation of f_n, θ_n is lengthy and somewhat involved and the reader is referred to the original papers for details. Here we shall simply examine the possibility of a satisfactory reconciliation between the numerical solution and the representations outlined in those papers. In particular we examine the implications with regards to the basic flow parameters, namely the skin friction and heat transfer coefficients. Near separation, for skin-friction coefficient correlation

$$\tau_w = \frac{1}{(2\bar{\xi})^{\frac{1}{2}}}(\bar{f}_{\zeta\zeta})_{\zeta=0} = 2^{\frac{1}{2}}\xi(f_{\eta\eta})_{\eta=0}$$
$$= 2^{\frac{1}{2}}\xi^2(2\alpha_{10}\ln\xi + 2\alpha_{11} + 2\alpha_{12}\ln|\ln\xi| + 2\alpha_{13}\ln|\ln\xi|/\ln\xi + \ldots) \tag{22}$$

and for heat transfer correlation

$$\bar{Q} = \frac{-1}{(2\bar{\xi})^{\frac{1}{2}}}(\bar{\theta}_{\zeta})_{\zeta=0} = \frac{-1}{2^{\frac{1}{2}}\xi}(\theta_\eta)_{\eta=0}$$
$$= \frac{-1}{2^{\frac{1}{2}}}\{b_1 - \xi K_2'(0)\,b_1(2\alpha_{10}\ln\xi + 2\alpha_{11} + 2\alpha_{12}\ln|\ln\xi| + 2\alpha_{13}\ln|\ln\xi|/\ln\xi + \ldots)\} \tag{23}$$

where

$$\alpha_{10} = \frac{-2\pi^{\frac{1}{2}}(-\frac{1}{4})!\,b_1}{64(\frac{1}{4}!)^3}, \quad \alpha_{12} = (1-2\ln 2)\,\alpha_{10},$$
$$\alpha_{13} = \frac{-64(\frac{1}{4}!)^3\,\alpha_{12}^2}{2\pi^{\frac{1}{2}}(-\frac{1}{4})!\,b_1}, \quad K_2'(0) = \frac{-2^{\frac{1}{4}}\pi^{\frac{3}{2}}}{8(\frac{1}{4}!)^3} \quad \text{(see appendix).} \tag{24}$$

The formulation implies $\xi = (\bar{\xi}_s - \bar{\xi})^{\frac{1}{4}}$, where $\bar{\xi}_s$ denotes the separation value of $\bar{\xi}$. Knowledge of the left-hand sides of (22) and (23) is available from the numerical integration. Of the two indeterminacies b_1, α_{11} the former may be specifically evaluated from the numerical solution at separation. On the other hand α_{11} is chosen to reconcile (22) and (23) over a range of ξ.

4. The constant heat flux case (*b*)

In non-dimensional terms the significant boundary condition for this case is $(\partial\theta'/\partial y')_{y'=0} = b_1 = 1$. Assuming, in the first instance, the power series representation of separation profiles, (15) remains appropriate whilst the counterpart of (16) reads

$$\theta' = b_0 + y' + b_2 y'^2 + b_3 y'^3 + \ldots. \tag{25}$$

Moreover the first compatibility condition for the absence of a singularity of (10) at $x' = 0$ now takes the slightly more general form

$$2a_2 + b_0 = 0. \tag{26}$$

To reflect the implications of (16) and (25), and yet allow for the possible arising of anomalies due to their assumed form, we follow Stewartson (1962) and seek solutions of (13) and (14) as

$$f(\xi,\eta) = \sum_{n=0}^{\infty} f_n(\eta)\,\xi^n + \text{extra terms involving } \ln\xi, \tag{27}$$

$$\theta(\xi,\eta) = \sum_{n=0}^{\infty} \theta_n(\eta)\,\xi^n + \text{extra terms involving } \ln\xi. \tag{28}$$

Appropriate boundary conditions are

$$f_n(0) = f_n'(0) = 0, \quad \lim_{\eta\to\infty} \frac{f_n'}{\eta^{n+2}} = 2^{\frac{1}{2}n} a_{n+2} \quad (n = 0, 1, 2, \ldots), \tag{29}$$

$$\theta_1'(0) = 2^{\frac{1}{2}}, \quad \theta_n'(0) = 0 \quad \text{for} \quad n \neq 1, \quad \lim_{\eta\to\infty} \frac{\theta_n}{\eta^n} = 2^{\frac{1}{2}} b_n \quad (n = 0, 1, 2, \ldots) \quad (\text{N.B.}\, b_1 \equiv 1), \tag{30}$$

unless it transpires that additional coefficient functions need to be introduced. Should this be the case then conditions as $\eta \to \infty$ are replaced by the requirement that coefficient function dependence on η should be algebraic for large η.

Equations for f_0, θ_0 are

$$f_0''' - 3f_0 f_0'' + 2f_0'^2 + \theta_0 = 0, \tag{31}$$

$$\theta_0'' - 3f_0 \theta_0' = 0. \tag{32}$$

The solution of (32) satisfying the boundary conditions is $\theta_0 \equiv b_0$. The solution for f_0 having a double zero at the origin and *satisfying the first compatibility condition* is

$$f_0 = \frac{-b_0 \eta^3}{6} = \frac{a_2 \eta^3}{3}. \tag{33}$$

Experience of the numerical solution suggests that a_2 may be expected to be positive.

Proceeding with the expansions (27) and (28) on the basis of (33) gives equations for f_1, θ_1 as

$$f_1''' - a_2 \eta^3 f_1'' + 5a_2 \eta f_1' - 8a_2 \eta f_1 = -\theta_1, \tag{34}$$

$$\theta_1'' - a_2 \eta^3 \theta_1' + a_2 \eta^2 \theta_1 = 0. \tag{35}$$

Equation (35) has as complementary functions η, and a function displaying exponential growth at large η. The required solution is therefore

$$\theta_1 = 2^{\frac{1}{2}} \eta. \tag{36}$$

With this solution for θ_1 the solution of (34) displaying a double zero at the origin and algebraic behaviour at large η is

$$f_1 = \alpha_1^* \eta^2 - 2^{\frac{1}{2}} \frac{\eta^4}{24}, \tag{37}$$

where α_1^* is the basic indeterminacy of the ensuing analysis. It is the counterpart of α_{11} of § 3. The additional subscript is not required here as the developing solution no longer displays the inconsistencies that Buckmaster had to account for. The asterisk is used to highlight this point. The significant stage at which distinction can be made between the two cases occurs when examining acceptable solution for f_2, θ_2. Their governing equations are

$$f_2''' - a_2 \eta^3 f_2'' + 6a_2 \eta^2 f_2' - 10a_2 \eta f_2 = -\theta_2 - \tfrac{1}{3}\alpha_1^* . 2^{\frac{1}{2}} . \eta^4 - 4\alpha_1^{*2} \eta^2, \tag{38}$$

$$\theta_2'' - a_2 \eta^3 \theta_2' + 2a_2 \eta^2 \theta_2 = 2^{\frac{3}{2}} \alpha_1^* \eta^2. \tag{39}$$

Note that, with the transformation $z = (2a_2)^{\frac{1}{4}} \eta$, the left-hand side operators of f_n, θ_n may be reduced to the forms dealt with in the appendix. Thus the complementary functions of (39) each display exponential behaviour at large η. They can therefore only appear in the solution as the combination [(constant) K_2]. However, since K_2' remains finite at $\eta = 0$, the boundary condition $\theta_2'(0) = 0$ leads to the conclusion that the constant multiplier must be identically zero. Accordingly, in contradistinction to the constant temperature case, K_2 does not appear in the solution for θ_2, which here is simply

$$\theta_2 = \frac{2^{\frac{1}{2}}\alpha_1^*}{a_2}. \tag{40}$$

The integral restraint on the right-hand side of (38) is now satisfied identically and an acceptable solution for f_2 is

$$f_2 = \alpha_2^*\eta^2 - \frac{2^{\frac{1}{2}}\alpha_1\eta^3}{6a_2} - \tfrac{1}{15}\alpha_1^{*2}\eta^5. \tag{41}$$

Although α_2^* is apparently arbitrary at this stage, it has the prescribed role of ensuring the absence of exponentially large terms in f_3. If a precise value of α_2^* is to be ascertained we must proceed to examine

$$f_3''' - a_2\eta^3 f_3'' + 7a_2\eta^2 f_3' - 12a_2\eta f_3 = -\theta_3 - 10\alpha_1^*\alpha_2^*\eta^2 + \frac{4}{3}\frac{2^{\frac{1}{2}}\alpha_1^{*2}\eta^3}{a_2} - \frac{2^{\frac{1}{2}}\alpha_2^*\eta^4}{2} - \frac{4\alpha_1^{*3}\eta^5}{3}, \tag{42}$$

$$\theta_3'' - a_2\eta^3\theta_3' + 3a_2\eta^2\theta_3 = 3.2^{\frac{1}{2}}.\alpha_2^*\eta^2 - \frac{2^{\frac{3}{2}}\alpha_1^{*2}\eta}{a_2}. \tag{43}$$

Again G_3, H_3 each display exponential behaviour at large η and only their appearance as the combination [(constant) K_3] can be countenanced. Once more the boundary condition $\theta_3'(0) = 0$, when applied to the general solution, requires that the constant be identically zero. The solution for θ_3 is

$$\theta_3 = \frac{2^{\frac{1}{2}}\alpha_2^*}{a_2} - \frac{2^{\frac{3}{2}}\alpha_1^{*2}\eta^3}{3a_2}. \tag{44}$$

As in the Goldstein analysis the solution for f_3 has to allow for two complementary functions g_3, h_3, each of which displays exponential behaviour at large η (see appendix). However a particular combination (in Terrill's notation k_3) may be shown to behave algebraically. Imposing the condition that g_3, h_3 appear only in such combination in

$$f_3 = \alpha_3^*\eta^2 + \frac{4\alpha_1^*\alpha_2^*}{(2a_2)^{\frac{3}{4}}}[z - g_3(z)] - \frac{8\alpha_1^{*3}}{3(2a_2)^2}\left[1 + \frac{z^4}{4} - h_3(z)\right] - \frac{2^{\frac{1}{2}}\alpha_2^*\eta^3}{3(2a_2)} + \frac{2^{\frac{1}{2}}\alpha_1^{*2}\eta^6}{30} \quad (z = (2a_2)^{\frac{1}{4}}\eta) \tag{45}$$

thus prescribes

$$\alpha_2^* = \frac{2^{\frac{1}{4}}\alpha_1^{*2}\pi^{\frac{3}{2}}}{(2a_2)^{\frac{3}{4}}\,5(\frac{1}{4}!)^3}. \tag{46}$$

In principle the solution may be pursued further and α_3^* obtained in like manner on examination of f_4. Unfortunately the presence of g_3, h_3 in (45) severely complicates subsequent developments. The possible occurrence of inconsistencies may, however, be closely monitored by due consideration of integral restraints associated with termination of complementary functions. For the energy equation these occur at ξ^{4m} and ξ^{4m+1} levels in the expansions and for the momentum equation at ξ^{4m+1} and ξ^{4m+2} levels. Accordingly the first likely source of complication would arise at the ξ^5 level in the energy equation. A contribution from the complementary function at the ξ^4 level which satisfies the zero derivative boundary condition at the wall together with algebraic behaviour at large η may be needed to allow satisfaction of the integral restraint at the ξ^5 level. Clarification of this point would however be extremely arduous and is beyond the scope of this present paper. We proceed therefore to examine the level of agreement between the solutions as far as f_2, θ_2, and the numerical solution. Again we shall concentrate on the basic flow parameters of skin friction and heat transfer coefficients. Near separation, for skin friction coefficient correlation, we require

$$
\begin{aligned}
\tau_w &= \frac{1}{(2\tilde{\xi})^{\frac{1}{2}}}(\tilde{f}_{\zeta\zeta})_{\zeta=0} = 5^{\frac{1}{3}}\xi(f_{\eta\eta})_{\eta=0} \\
&= 5^{\frac{1}{3}}\{2\alpha_1^*\xi^2 + 2\alpha_2^*\xi^3 + .\} \\
&= 5^{\frac{1}{3}}\left\{2\alpha_1^*\left[\frac{5^{\frac{2}{3}}}{2}(\tilde{\xi}_s - \tilde{\xi})\right]^{\frac{1}{2}} + 2\alpha_2^*\left[\frac{5^{\frac{2}{3}}}{2}(\tilde{\xi}_s - \tilde{\xi})\right]^{\frac{3}{4}} + \ldots\right\}
\end{aligned}
\tag{47}
$$

and for heat transfer coefficient correlation

$$
\begin{aligned}
\tilde{Q} &= \frac{-5^{\frac{1}{3}}}{2^{\frac{1}{2}}(\theta)_{\eta=0}} \\
&= \frac{-5^{\frac{1}{3}}}{2^{\frac{1}{2}}}\left\{b_0 + \frac{2^{\frac{1}{2}}\alpha_1^*\xi^2}{a_2} + \frac{2^{\frac{1}{2}}\alpha_2^*\xi^3}{a_2} + \ldots\right\} \\
&= \frac{-5^{\frac{1}{3}}}{2^{\frac{1}{2}}}\left\{b_0 + \frac{2^{\frac{1}{2}}\alpha_1^*}{a_2}\left[\frac{5^{\frac{2}{3}}}{2}(\tilde{\xi}_s - \tilde{\xi})\right]^{\frac{1}{2}} + \frac{2^{\frac{1}{2}}.\alpha_2^*}{a_2}\left[\frac{5^{\frac{2}{3}}}{2}(\tilde{\xi}_s - \tilde{\xi})\right]^{\frac{3}{4}} + \ldots\right\}.
\end{aligned}
\tag{48}
$$

The left-hand sides of (47) and (48) are known from the numerical integration. The temperature at the wall at separation b_0 is specifically evaluated by the numerical integration whereas α_1^* is chosen to reconcile the analytic and the numerical solution near separation. Note that (47) and (48) provide two independent means of estimating α_1^*.

5. Numerical procedure

The equations to be solved (Merkin 1969; Wilks 1974) are

$$
\left.
\begin{aligned}
\frac{\partial^3 f}{\partial\zeta^3} + f\frac{\partial^2 f}{\partial\zeta^2} + \lambda\theta + 2\xi'\left(\frac{\partial^2 f}{\partial\zeta^2}\frac{\partial f}{\partial\xi'} - \frac{\partial f}{\partial\zeta}\frac{\partial^2 f}{\partial\xi'\partial\zeta}\right)\partial\zeta &= 0, \\
\frac{1}{P_r}\frac{\partial^2\theta}{\partial\zeta^2} + f\frac{\partial\theta}{\partial\zeta} - \mu\theta\frac{\partial f}{\partial\zeta} + 2\xi'\left(\frac{\partial\theta}{\partial\zeta}\frac{\partial f}{\partial\xi'} - \frac{\partial f}{\partial\zeta}\frac{\partial\theta}{\partial\xi'}\right) &= 0.
\end{aligned}
\right\}
\tag{49}
$$

where for case (*a*) $\xi' = \bar{\xi}$, $\lambda = -2$, $\mu = 0$, and case (*b*) $\xi' = \tilde{\xi}$, $\lambda = 5\tilde{\xi}^{\frac{3}{2}}$, $\mu = 1$; subject to boundary conditions

$$
f = \frac{\partial f}{\partial\zeta} = 0, \quad \left\{\begin{matrix} (a) & \theta = 1 \\ (b) & \dfrac{\partial\theta}{\partial\zeta} = 1 \end{matrix}\right\} \quad \text{on} \quad \zeta = 0
$$

$$
\frac{\partial f}{\partial\zeta} \to 1, \quad \theta \to 0 \quad \text{as} \quad \zeta \to \infty.
$$

These equations have been solved numerically using a method devised by Keller (Keller & Cebeci 1971; Keller 1978). This method has advantages over the method used previously (Terrill 1960) in that the solution at $\xi' = 0$ is readily calculated, the method is unconditionally stable and principally that the method allows us to employ Richardson's extrapolation enabling us to obtain high accuracy using crude nets.

The equations are firstly recast into linear form by introducing the variables u, v and w defined as

$$\frac{\partial f}{\partial \zeta} = u, \quad \frac{\partial u}{\partial \zeta} = v, \quad \frac{\partial \theta}{\partial \zeta} = w. \tag{50a, b, c}$$

The equations now read

$$\frac{\partial v}{\partial \eta} + fv + \lambda\theta + 2\xi'\left\{u\frac{\partial u}{\partial \xi'} - v\frac{\partial f}{\partial \xi'}\right\} = 0, \tag{51a}$$

$$\frac{1}{P_r}\frac{\partial w}{\partial \eta} + fw - \mu\theta w + 2\xi'\left\{u\frac{\partial \theta}{\partial \xi'} - w\frac{\partial f}{\partial \xi'}\right\} = 0, \tag{51b}$$

and boundary conditions lead to

$$\left.\begin{array}{l} f = 0, \quad u = 0, \quad \begin{Bmatrix}(a) \ \ \theta = 1 \\ (b) \ \ w = 1\end{Bmatrix} \quad \text{at} \quad \eta = 0 \\ \qquad u = 1, \quad \theta = 0 \quad \text{at} \quad \eta = \infty. \end{array}\right\} \tag{52}$$

To discretize the equations we use a net which is non-uniform in ξ' but uniform in η, defined as

$$\left.\begin{array}{l} \xi_0' = 0, \quad \xi_n' = \xi_{n-1}' + k_n \quad (n = 1, 2, \ldots), \\ \zeta_0 = 0, \quad \zeta_j = \zeta_{j-1} + h \quad (j = 1, 2, \ldots, N), \end{array}\right\} \tag{53}$$

where the outer boundary ζ_N has been set at 7·2. If g_j^n denotes the value of any variable g at (ξ_n', η_j), then variables and derivatives of equations (51) at $(\xi_{n-\frac{1}{2}}', \eta_{j-\frac{1}{2}})$ are replaced by

$$g_{j-\frac{1}{2}}^{n-\frac{1}{2}} = \tfrac{1}{4}(g_j^n + g_{j-1}^n + g_j^{n-1} + g_{j-1}^{n-1}),$$

$$\left[\frac{\partial g}{\partial \xi'}\right]_{j-\frac{1}{2}}^{n-\frac{1}{2}} = \frac{1}{2k_n}(g_j^n + g_{j-1}^n - g_j^{n-1} - g_{j-1}^{n-1}),$$

$$\left[\frac{\partial g}{\partial \zeta}\right]_{j-\frac{1}{2}}^{n-\frac{1}{2}} = \frac{1}{2h}(g_j^n + g_j^{n-1} - g_{j-1}^n - g_{j-1}^{n-1}),$$

where $\xi_{n-\frac{1}{2}}' = \xi_{n-1}' + \frac{1}{2}k_n$ and $\eta_{j-\frac{1}{2}} = \eta_{j-1} + \frac{1}{2}h$. Equations (50) are centred at $(\xi_n', \eta_{j-\frac{1}{2}})$ and we therefore use

$$g_{j-\frac{1}{2}}^n = \tfrac{1}{2}(g_j^n + g_{j-1}^n), \quad \left[\frac{\partial g}{\partial \zeta}\right]_{j-\frac{1}{2}}^n = \frac{1}{h}(g_j^n - g_{j-1}^n). \tag{54}$$

The boundary conditions (52) are then

$$f_0^n = 0, \quad u_0^n = 0, \quad \begin{Bmatrix}(a) \ \ \theta_0^n = 1, \\ (b) \ \ w_0^n = 1,\end{Bmatrix} \quad u_N^n = 1, \quad \theta_N^n = 0.$$

If we suppose we have solved the problem up to ξ_{n-1}, then we have $5N$ equations plus 5 boundary conditions for the $5N+5$ unknowns $(f_j^n, u_j^n, v_j^n, \theta_j^n, w_j^n)$ $j = 0, 1, \ldots, N$. These are nonlinear algebraic equations which are solved using Newton's iteration, the values of the variables at ξ_{n-1}' being using as an initial iterate. At $\xi' = 0$ the equations (51) have only η derivatives and are discretized using (54). The resulting algebraic equations are again solved by Newton's iteration.

As we approach separation (i.e. $v = 0$), the value of k_n is determined from the two most recent values of v at $\eta = 0$, namely v_0^{n-1} and v_0^{n-2}. If we assume that v_0^n is approximately proportional to $(\xi_s' - \xi_n')^{\frac{1}{2}}$ as we approach separation (where ξ_s' is the position of separation) then we can estimate ξ_s' from v_0^{n-1} and v_0^{n-2}, and after some algebra we find

$$\xi_s' - \xi_{n-1}' = \frac{k_{n-1}(v_0^{n-1})^2}{(v_0^{n-2})^2 - (v_0^{n-1})^2}.$$

Hence by choosing k_n to be $\frac{1}{2}(\xi_s' - \xi_{n-1}')$ as given by this estimate, one is able to approach separation by continuously halving the distance to separation.

Each cell of the net (53) is divided into $2m$ equal subintervals in the ξ direction and m subintervals in η producing a finer net having cell dimensions $k_n/2m$ and h/m where m is an integer. The program was run for values of $m = 2$, 3 and 4, having set $N = 10$, and Richardson's extrapolation is employed in order to obtain results of higher accuracy. Since the truncation error is a power series in the square of $k/2m$ and h/m (where $k = \max_n k_n$) the final result will have truncation error $O(h^6 + k^6)$. It should be appreciated that the calculated separation point ξ_s' will have error $O((h/2m)^2 + (k/m)^2)$, and again Richardson's extrapolation is used to find a more accurate value. The intermediate values ξ_n' will similarly vary and are treated in an analogous manner. The values of ξ_s' given by Merkin (1969) and Wilks (1974) contain an error $O(h^2)$ and differ from our results, which being $O(h^6)$ are substantially more accurate.

In order to assess accuracy a further run was made setting $m = 1$. The results for $m = 1, 2$ and 3 were used to obtain a further set of results $O(h^6 + k^6)$ which were used to test the accuracy of the original set.

6. Numerical results

The numerical results obtained are accurate to approximately 6 decimal places except for the temperature gradient w in case (a) within 10^{-4} of separation (at distances of 10^{-5} to 10^{-4} to separation the accuracy in w is about 3 or 4 decimal places). Because of the exponential growth in the error of w as separation is approached in case (a) the program failed to converge at distances less than 5×10^{-6} to separation. However in case (b) no such difficulty is encountered and in fact separation was approached to within 10^{-8}.†

Tables 1 and 2 show the values of the flow parameter for values of ξ' up to separation for cases (a) and (b) respectively, where $\tau_w = (v)_{\eta=0}/\sqrt{(2\xi')}$ is the skin friction coefficient (column 2) and $\bar{Q} = -(w)_{\eta=0}/\sqrt{(2\bar{\xi})}$ and $\tilde{Q} = -1/((\theta)_{\eta=0}\sqrt{(2\tilde{\xi})})$ are the heat transfer coefficients for cases (a) and (b) respectively (column 4). The separation points ξ_s' are estimated as the point where an extrapolation of τ_w^2 becomes zero and are found to be

$$(a)\ \bar{\xi}_s = 0{\cdot}192\,217, \qquad (b)\ \tilde{\xi}_s = 0{\cdot}141\,576\,99,$$

and are accurate to the number of decimal places quoted as are all results in this paper. At this point we find

$$(a)\ \bar{Q} \to 0{\cdot}423, \qquad (b)\ \tilde{Q} \to 0{\cdot}952\,068.$$

† There would be no difficulty in getting closer.

$\bar{\xi}$	τ_w	τ_w (series)	$\bar{Q}$	$\bar{Q}$ (series)
0·040 000	1·422 990	0·484 71	1·601 885	0·680 8
0·063 220	1.015 370	0.439 69	1.243 449	0.666 7
0·085 005	0.776 127	0.394 84	1.044 480	0.652 2
0·104 984	0.611 073	0.350 67	0.913 782	0.637 3
0·122 953	0·486 878	0·307 48	0·819 449	0·622 0
0·138 726	0·388 808	0·265 68	0·747 395	0·606 5
0·152 162	0·309 331	0·225 83	0·690 355	0·590 6
0·163 205	0·244 219	0·188 59	0·644 185	0·574 7
0·171 917	0·190 889	0·154 61	0·606 334	0·559 0
0·178 490	0·147 573	0·124 46	0·575 115	0·543 7
0·183 227	0·112 860	0·098 48	0·549 320	0·529 1
0·186 495	0·085 472	0·076 74	0·528 012	0·515 6
0·188 662	0·064 194	0·059 01	0·510 424	0·503 2
0·190 053	0·047 888	0·044 89	0·495 910	0·492 0
0·190 920	0·035 533	0·033 84	0·483 929	0·482 1
0·191 450	0·026 255	0·025 32	0·474 026	0·473 4
0·191 768	0·019 337	0·018 84	0·465 829	0·465 9
0·191 956	0·014 206	0·013 95	0·459 030	0·459 3
0·192 067	0·010 416	0·010 29	0·453 380	0·453 7
0·192 131	0·007 626	0·007 56	0·448 670	0·448 9
0·192 168	0·005 577	0·005 54	0·444 737	0·444 8
0·192 189	0·004 074	0·004 04	0·441 450	0·441 3
0·192 201	0·002 974	0·002 94	0·438 698	0·438 3
0·192 208	0·002 166	0·002 18	0·436 562	0·436 0
0·192 212	0·001 577	0·001 60	0·434 765	0·433 9

TABLE 1

The flow parameters τ_w and $\bar{Q}$ for case (*a*) were matched to the series solution at separation (equations (22) and (23)) to determine the parameters α_{11} and b_1. For any given $\bar{\xi}$ one may use (22) and (23) to obtain these parameters and if the series solution and the numerical solution are consistent then the values obtained for α_{11} and b_1 should not vary with $\bar{\xi}$. This is indeed the case, thus the numerical results confirm the validity of the series obtained. Choosing the parameters such that (22) has error $O((\bar{\xi}_s-\bar{\xi})^{\frac{3}{4}})$ and (23) $O((\bar{\xi}_s-\bar{\xi})^{\frac{1}{2}})$ gave

$$\alpha_{11} = 0{\cdot}436, \quad b_1 = -0{\cdot}598.$$

Using these values gave series solution values for τ_w and $\bar{\theta}$ as shown in columns (3) and (5) of table 1. Notice that b_1 is negative, consistent with temperature decrease away from the wall. Nevertheless very satisfactory matching is achieved in the context of the Buckmaster theory despite the implications that this has on the skin friction estimates extremely close to separation. Similar paradoxical conclusions were reached by Davies & Walker for hot walls. Moreover, if the flow envisaged is converted to its counterpart of a cold wall in a heated stream, exactly the same equations and results obtain. Note that in this latter case $b_1 < 0$ is not inconsistent with temperature increase away from the wall since the temperature decreases or increases in accordance with the positive or negative nature of $(T_1 - T_0)$.

In a similar fashion the parameters α_1^* and b_0 of equations (47) and (48) for case (*b*) can be determined using τ_w and $\tilde{Q}$ in table 2. It was found that b_0 could be assessed very

$\tilde{\xi}$	τ_w	τ_w (series)	$\tilde{Q}$	$\tilde{Q}$ (series)
0·040 000 00	1·509 968	0·876 311	2·261 140	1·728 940
0·068 208 28	1·003 261	0·727 287	1·695 565	1·518 258
0·088 918 91	0·746 687	0·602 242	1·452 104	1·377 421
0·105 005 96	0·568 585	0·490 215	1·304 720	1·271 732
0·117 166 08	0·433 492	0·390 900	1·205 296	1·190 735
0·125 928 94	0·328 306	0·305 360	1·135 105	1·128 812
0·131 917 32	0·246 379	0·234 141	1·084 599	1·081 966
0·135 806 52	0·183 237	0·176 758	1·048 016	1·046 958
0·138 222 88	0·135 216	0·131 799	1·021 478	1·021 073
0·139 670 67	0·099 141	0·097 337	1·002 229	1·002 082
0·140 513 40	0·072 307	0·071 353	0·988 274	0·988 224
0·140 992 66	0·052 505	0·051 999	0·978 164	0·978 148
0·141 260 08	0·037 988	0·037 720	0·970 851	0·970 845
0·141 407 01	0·027 402	0.027 258	0.965 566	0.965 564
0·141 486 72	0·019 711	0·019 632	0·961 752	0·961 750
0·141 529 45	0·014 144	0·014 102	0·959 005	0·959 002
0·141 552 13	0·010 130	0.010 107	0·957 030	0·957 028
0·141 564 07	0.007 242	0.007 229	0.955 612	0.955 611
0·141 570 32	0.005 170	0.005 163	0.954 596	0.954 595
0.141 573 56	0·003 685	0·003 681	0·953 870	0·953 868
0·141 575 23	0·002 624	0·002 620	0·953 350	0·953 349
0·141 576 10	0·001 865	0·001 863	0·952 979	0·952 978
0·141 576 54	0·001 327	0·001 324	0·952 716	0·952 715
0·141 576 76	0·000 940	0·000 939	0·952 527	0·952 527
0·141 576 87	0·000 672	0·000 667	0·952 396	0·952 394
0·141 576 93	0·000 472	0·000 466	0·952 299	0·952 296
0·141 576 96	0·000 331	0·000 329	0·952 230	0·952 229
0·141 576 98	0·000 241	0·000 239	0·952 186	0·952 185
0·141 576 98	0·000 166	0·000 166	0·952 149	0·952 149
0·141 576 99	0·000 123	0·000 122	0·952 128	0·952 128

TABLE 2

accurately by using (47) to eliminate α_1^* and α_2^* from (48) and considering small values of $(\tilde{\xi}_s - \tilde{\xi})$. It was found that

$$b_0 = -1{\cdot}270\,010.$$

Having determined b_0, α_1^* could be assessed either from τ_w or $\tilde{Q}$ at each position $\tilde{\xi}$. It was found that α_1^* was remarkably constant with $\tilde{\xi}$ and gave the same result from both τ_w and $\tilde{Q}$, corroborating the series solution to a high degree of accuracy; α_1^* was determined such that the error in (47) and (48) is $O(\tilde{\xi}_s - \tilde{\xi})$ and is

$$\alpha_1^* = 0{\cdot}4653 \text{ (using } \tau_w), \quad \alpha_1^* = 0{\cdot}4651 \text{ (using } \tilde{Q}).$$

Taking α_1^* to be 0.4652 and b_0 as given gave series solution values for τ_w and $\tilde{Q}$, see table 2.

7. Conclusion

It has been demonstrated that a clear distinction exists between the singularities in mixed convection boundary-layer separation associated with the uniform temperature and uniform heat flux boundary condition respectively. The relatively straightforward asymptotic structure about the singularity occurring in the uniform heat flux case demonstrates that Buckmaster expansions are not an inherent feature of the coupling of the momentum and energy equations. This same relative straightforwardness recommends the uniform heat flux case as the natural one to examine, in the first instance, when attempts are made subsequently to embed the analysis evidenced here into an investigation of separation in the context of the full Navier–Stokes equations.

Appendix

Complementary functions play a significant role in developing consistent series solutions. We present here a summary of the complementary functions and their properties, associated with the operators arising from the momentum and energy equations.

The homogeneous differential equation whose solutions are relevant to the stream functions f_n is

$$\dddot{f}_n - \tfrac{1}{2}z^3\ddot{f}_n + \tfrac{1}{2}(n+4)\,z^2\dot{f}_n - (n+3)\,zf_n = 0,$$

where the dots indicate differentiation with respect to z. Complementary functions are z^2, $h_n(z)$, $g_n(z)$ (see Goldstein 1948); $h_n(0) = 1, g_n(0) = 0$; $\dot{h}_n(0) = 0$, $\dot{g}_n(0) = 1$. When $n = 4m+1$ the series representation of $h_n(z)$ terminates, whilst that of $g_n(z)$ terminates when $n = 4m+2$, for m a positive integer or zero. Otherwise $h_n(z)$, $g_n(z)$ each display exponential behaviour at large z. However when $n \neq 4m+1$, $4m+2$ the combination

$$k_n = h_n + \frac{2^{\frac{1}{4}}(-\frac{5}{4})!\,(-\frac{3}{2}-\frac{1}{4}n)!}{(-\frac{3}{4})!\,(-\frac{7}{4}-\frac{1}{4}n)!}\,g_n$$

behaves algebraically for large z and has the value unity at the origin.

In the development of the temperature function θ_n the relevant equation is

$$\ddot{\theta}_n - \tfrac{1}{2}z^3\dot{\theta}_n + \tfrac{1}{2}nz^2\theta_n = 0.$$

Series solutions are

$$H_n(z) = M\left(-\frac{n}{4}, \frac{3}{4}; \frac{z^4}{8}\right), \quad G_n(z) = zM\left(\frac{1-n}{4}, \frac{5}{4}; \frac{z^4}{8}\right),$$

where $H_n(0) = 1, \dot{H}_n(0) = 0; G_n(0) = 0, \dot{G}_n(0) = 1$ and $M(a,b;z)$ is Kummer's confluent hypergeometric function. The series representation of $H_n(z)$ terminates when $n = 4m$ whilst that of $G_n(z)$ terminates when $n = 4m+1$. Otherwise $H_n(z)$, $G_n(z)$ each display exponential behaviour at large Z. When $n \neq 4m$, $4m+1$ the combination

$$K_n = H_n - \frac{(-\frac{1}{4})!\,(-\frac{1}{4}n-\frac{3}{4})!}{(\frac{1}{4})!\,(-1-\frac{1}{4}n)!\,8^{\frac{1}{4}}}\,G_n$$

behaves algebraically for large z and has the value unity at the origin.

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